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Periodic Solutions of Complex-Valued Differential Equations and Systems with Periodic Coefficients*

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1. INTRODUCTION

The existence of periodic solutions of complex valued ordinary differential equations of the form

$$z' = \sum_{0 \leqslant k+l < r+s} b_{kl}(t)\, z^k \bar{z}^l + c e^{imt} z^r \bar{z}^s$$

has been considered by Srzednicki [11–13], using a combination of generalized Conley's isolating blocks, the index of singular points of autonomous systems and the Lefschetz fixed point theorem.

Recently, it has been shown that slight extensions of Srzednicki's results could be obtained in a simpler and more elementary way using continuation theorems of the Leray–Schauder type (see [6, 9]). The aim of the present paper is to show that the same methodology can be adapted to prove the existence of periodic solutions for more general classes of equations and systems.

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In Section 2, we deal with systems of equations of the form

$$a_i(t)\, z_i' = |z_i|^q\, \bar{z}_i^p + h_i(t, z), \qquad (1 \leqslant i \leqslant n),$$

with $z = (z_1, ..., z_n)$, when

$$|h_i(t, z)| \leqslant \sum_{j=1}^{n} \alpha_{ij}\, |z_j|^{p+q} + \gamma_i, \qquad (1 \leqslant i \leqslant n),$$

for some sufficiently small $\alpha_{ij} \geqslant 0$ and some $\gamma_i \geqslant 0$. Here each a_i: $[0, 2\pi] \to \mathbb{C} \backslash \{0\}$ is supposed to be 2π-periodic, $q \geqslant 0$ is a real number and $p \geqslant 1$ an integer. Theorem 1 insures that a 2π-periodic solution exists when $p + q > 1$. A counter-example shows that this condition cannot be weakened to $p + q = 1$ (i.e., $q = 0, p = 1$). The proof of Theorem 1 is based upon a Leray–Schauder continuation argument, with a homotopy to a linear system obtained *a posteriori* in order to make possible the obtention of a priori bounds in a suitable integral norm. Interestingly, this linear system is critical or noncritical according to $p + 1$ divides or not the variation of the argument of some a_i over one period.

The case of systems of the form

$$a_i(t)\, z_i' = |z_i|^q\, z_i^p + h_i(t, z), \qquad (1 \leqslant i \leqslant n),$$

with $q \geqslant 0$ and $p > 1$, seems more difficult. Indeed, it contains as a special case the Riccati equation

$$z' = z^2 + g(t)\, z + f(t),$$

which may have no periodic solutions for some choices of f and g as follows from results of Lloyd [5], Hassan [4], and Campos and Ortega [2]. Existence theorems can however be obtained in the special case where $p = 1$, i.e., in the case of a system with a *Ginzburg–Landau's nonlinearity*

$$a_i(t)\, z' = |z_i|^q\, z_i + h_i(t, z), \qquad (1 \leqslant i \leqslant n),$$

with the a_i and h_i continuous, and are given in Section 3. For such systems, using Krasnosel'skii's method of guiding functions, the existence of a 2π-periodic solution is proved in Theorem 2 under the assumption that all the functions $\Re a_i(t)$ never vanish and have the same sign, and that h satisfies a condition which holds in particular when

$$\lim_{|z| \to \infty} \frac{h(t, z)}{|z|^{q+1}} = 0,$$

uniformly in $t \in [0, 2\pi]$, where $|z| = (\sum_{i=1}^{n} |z_i|^2)^{1/2}$. In the same section, an argument similar to that of Theorem 1 is used in Theorem 3 to obtain the existence of periodic solutions in the case of C^1 and 2π-periodic functions a_i such that the $\Re a_i$ may have different signs.

Notice that the systems considered in this paper respectively occur in spatial discretizations of nonlinear Ginsburg–Landau and Schrödinger partial differential equations like

$$\alpha(t, x) \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + |z|^q \bar{z}^p + \varphi(t, x),$$

and

$$\alpha(t, x) \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + |z|^q z + \varphi(t, x).$$

For the reader's convenience, we recall next the abstract continuation theorem of [7] used in proving Theorem 1 and 3, and which provides also, as shown in [7], an easy proof of Krasnosel'skii's theorem used in Theorem 2. Let X and Z be real normed spaces, with the open ball centered at zero and of radius r denoted by $B(r)$. Let $L: D(L) \subset X \to Z$ be a linear mapping and $N: X \to Z$ be a possibly nonlinear one. The following lemma is, up to notations, a special case of Corollary IV.7 and Theorem IV.13 of [7].

Lemma 1. *Assume that there exists a linear continuous mapping $A: X \to Z$ such that $L - A$ is a Fredholm linear mapping with zero index and $N - A$ is $(L - A)$-completely continuous on X. Denote by $S: Z \to Z$ a continuous projector onto* $\operatorname{Im}(L - A)$ *and write $Q = I - S$. Suppose there is some $R > 0$ such that, for each $\lambda \in\,]0, 1[$ and each possible solution of the equation*

$$Lu = (1 - \lambda) Au + \lambda Nu,$$

one has $\|u\| < R$. Then, if either

$$\ker(L - A) = \{0\}$$

or

$$Q(N - A)v \neq 0$$

for each $v \in \ker(L - A)$ with $\|v\| \geqslant R$ and the Brouwer degree

$$\deg_B[Q(N - A)|_{\ker(L - A)}, B(R) \cap \ker(L - A), 0]$$

is different from zero, then the equation

$$Lu = Nu \tag{1}$$

has at least one solution u *such that* $\|u\| \leqslant R$.

2. A Class of Complex Valued Systems

We consider the periodic boundary value problem

$$a_i(t)\, z_i' = |z_i|^q\, \bar{z}_i^p + h_i(t, z), \qquad z_i(0) - z_i(2\pi) = 0, \qquad (1 \leqslant i \leqslant n) \tag{2}$$

where $q \in \mathbb{R}_+$ and $p \geqslant 1$, $n \geqslant 1$ are integers, and $z = (z_1, ..., z_n)$. We assume that the functions h_i: $[0, 2\pi] \times \mathbb{C}^n \to \mathbb{C}$ are continuous and the a_i: $[0, 2\pi] \to \mathbb{C} \backslash \{0\}$ of class C^1 and $2\pi -$periodic, $(i = 1, 2, ..., n)$.

Our aim in this section is to prove the following existence result.

Theorem 1. *Assume that* $p + q > 1$ *and that*

$$|h_i(t, z)| \leqslant \sum_{j=1}^{n} \alpha_{ij}\, |z_j|^{p+q} + \gamma_i, \qquad (1 \leqslant i \leqslant n) \tag{3}$$

for some nonnegative numbers α_{ij}, γ_i, $(1 \leqslant i, j \leqslant n)$. *If the matrix*

$$\begin{pmatrix} 1 - \sum_{j=1}^{n} \alpha_{1j} & -\alpha_{12} & -\alpha_{13} & \cdots & -\alpha_{1n} \\ -\alpha_{21} & 1 - \sum_{j=1}^{n} \alpha_{2j} & -\alpha_{23} & \cdots & -\alpha_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1} & -\alpha_{n2} & -\alpha_{n3} & \cdots & 1 - \sum_{j=1}^{n} \alpha_{nj} \end{pmatrix} \tag{4}$$

is positive definite, then the problem (2) *has at least one solution.*

Proof. Problem (2) can be written in the form (1), where

$$X = Z = C([0, 2\pi], \mathbb{C}^n),$$

with the usual uniform norm,

$$D(L) = \{z \in X : z \text{ is of class } C^1 \text{ and } z(0) = z(2\pi)\}, \qquad Lz = z', \tag{5}$$

$$Nz = \left(\frac{1}{a_1} [\,|z_1|^q\, \bar{z}_1^p + h_1(\cdot, z)], ..., \frac{1}{a_n} [\,|z_n|^q\, \bar{z}_n^p + h_n(\cdot, z)] \right). \tag{6}$$

We use a continuation argument based upon Lemma 1 and try to determine continuous functions b_i: $\mathbb{R} \to \mathbb{C}$, $(1 \leqslant i \leqslant n)$ such that an a priori bound exists for the possible solutions of the one-parameter family of problems

$$a_i(t)\, z_i' + b_i(t)\, z_i = \lambda[b_i(t)\, z_i + h_i(t, z) + |z_i|^q\, \bar{z}_i^p], \qquad \lambda \in\,]0, 1],$$

$$z_i(0) = z_i(2\pi), \qquad (1 \leqslant i \leqslant n).$$

If z is a possible solution of this system for some $\lambda \in\,]0, 1]$, then, multiplying both members of the ith equation by z_i^p, we obtain

$$\frac{d}{dt}\left[\frac{a_i(t)\, z_i^{p+1}}{p+1}\right] + \left[b_i(t) - \frac{a_i'(t)}{p+1}\right] z_i^{p+1}$$
$$= \lambda[b_i(t)\, z_i^{p+1} + |z_i|^{2p+q} + z_i^p h_i(t, z)]. \tag{7}$$

If we now choose

$$b_i(t) = \frac{a_i'(t)}{p+1},$$

i.e., if we consider the homotopy

$$a_i(t)\, z_i' + (1-\lambda)\frac{a_i'(t)}{p+1}\, z_i = \lambda[\, |z_i|^q\, \bar{z}_i^p + h_i(t, z)], \qquad \lambda \in\,]0, 1], \tag{8}$$

$$z_i(0) = z_i(2\pi), \qquad (1 \leqslant i \leqslant n),$$

and if we integrate both members of (7) over $[0, 2\pi]$, we get, after dividing by $\lambda > 0$,

$$\|z_i\|^{2p+q} \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left[\, |h_i(t, z)\, z_i^p| + \frac{|a_i'(t)|}{p+1}\, |z_i(t)|^{p+1}\right] dt, \qquad (1 \leqslant i \leqslant n), \tag{9}$$

where

$$\|z_i\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |z_i(t)|^{2p+q}\, dt\right)^{1/(2p+q)}, \qquad (1 \leqslant i \leqslant n).$$

Using assumption (3), we deduce from (9) that

$$\|z_i\|^{2p+q} \leqslant \sum_{j=1}^{n}\left(\frac{1}{2\pi}\int_0^{2\pi} \alpha_{ij}\, |z_j(t)|^{p+q}\, |z_i(t)|^p\, dt\right) + \frac{\gamma_i}{2\pi}\int_0^{2\pi} |z_i(t)|^p\, dt$$
$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{|a_i'(t)|}{p+1}\, |z_i(t)|^{p+1}\, dt \qquad (1 \leqslant i \leqslant n),$$

and hence, letting $\eta_i = \sup_{t\in[0,2\pi]} |a_i'(t)|$ $(1 \leqslant i \leqslant n)$, and using Hölder inequality, we get

$$(1-\alpha_{ii})\,\|z_i\|^{2p+q} \leqslant \sum_{j=1,\, j\neq i}^{n} \frac{\alpha_{ij}}{2\pi}\int_0^{2\pi} |z_j(t)|^{p+q}\,|z_i(t)|^p\,dt$$
$$+\gamma_i\,\|z_i\|^p + \frac{\eta_i}{p+1}\,\|z_i\|^{p+1}, \qquad (1\leqslant i\leqslant n). \tag{10}$$

Now, letting $\Omega_{ij} = \{t\in[0,\pi] : |z_i(t)| < |z_j(t)|\}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |z_j(t)|^{p+q}\,|z_i(t)|^p\,dt$$
$$= \frac{1}{2\pi}\int_{\Omega_{ij}} |z_j(t)|^{p+q}\,|z_i(t)|^p\,dt + \frac{1}{2\pi}\int_{[0,2\pi]\setminus\Omega_{ij}} |z_j(t)|^{p+q}\,|z_i(t)|^p\,dt$$
$$\leqslant \frac{1}{2\pi}\int_{\Omega_{ij}} |z_j(t)|^{2p+q}\,dt + \frac{1}{2\pi}\int_{[0,2\pi]\setminus\Omega_{ij}} |z_i(t)|^{2p+q}\,dt$$
$$\leqslant \|z_j\|^{2p+q} + \|z_i\|^{2p+q}, \qquad (1\leqslant i\neq j\leqslant n). \tag{11}$$

Therefore, it follows from (10) that, for each $1\leqslant i\leqslant n$,

$$\left(1-\sum_{j=1}^{n}\alpha_{ij}\right)\|z_i\|^{2p+q} \leqslant \sum_{j=1,\, j\neq i}^{n} \alpha_{ij}\,\|z_j\|^{2p+q} + \gamma_i\,\|z_i\|^p + \frac{\eta_i}{p+1}\,\|z_i\|^{p+1}.$$

Consequently, the assumption upon matrix (4) implies the existence of $\mu>0$ such that

$$\mu\sum_{j=1}^{n}\|z_j\|^{2(2p+q)} \leqslant \sum_{j=1}^{n}\left[\gamma_j\,\|z_j\|^{3p+q} + \frac{\eta_j}{p+1}\,\|z_j\|^{3p+q+1}\right].$$

Using Hölder inequality for sums, we deduce from this inequality that

$$\mu\sum_{j=1}^{n}\|z_j\|^{2(2p+q)}$$
$$\leqslant \left(\sum_{j=1}^{n}\gamma_j^{2(2p+q)/(p+q)}\right)^{(p+q)/2(2p+q)} \left(\sum_{j=1}^{n}\|z_j\|^{2(2p+q)}\right)^{(3p+q)/2(2p+q)}$$
$$+\frac{1}{p+1}\left(\sum_{j=1}^{n}\eta_j^{2(2p+q)/(p+q-1)}\right)^{(p+q-1)/2(2p+q)}$$
$$\times\left(\sum_{j=1}^{n}\|z_j\|^{2(2p+q)/(3p+q+1)}\right)^{(3p+q+1)/2(2p+q)}.$$

Therefore, letting

$$\|z\|_* = \left(\sum_{j=1}^{n} \|z_j\|^{2(2p+q)} \right)^{1/2(2p+q)},$$

$$\gamma = \left(\sum_{j=1}^{n} \gamma_j^{2(2p+q)/(p+q)} \right)^{(p+q)/2(2p+q)},$$

$$\eta = \left(\sum_{j=1}^{n} \eta_j^{2(2p+q)/(p+q-1)} \right)^{(p+q-1)/2(2p+q)},$$

we obtain

$$\mu \|z\|_*^{2(2p+q)} \leqslant \gamma \|z\|_*^{3p+q} + \frac{\eta}{p+1} \|z\|_*^{3p+q+1},$$

and hence

$$\|z\|_* \leqslant R_1,$$

where R_1 is the largest positive root of the algebraic equation

$$\mu v^{p+q} - \frac{\eta}{p+1} v - \gamma = 0.$$

Consequently

$$\|z_i\| \leqslant R_1, \qquad (1 \leqslant i \leqslant n). \tag{12}$$

Using those estimates (12), we easily show, using the equation and Hölder inequality again, that each possible solution of (8) satisfies an a priori bound of the form

$$\frac{1}{2\pi} \int_0^{2\pi} |z_i'(t)| \, dt \leqslant R_2, \qquad (1 \leqslant i \leqslant n),$$

and hence there exists $R > 0$ such that each possible solution of (8) satisfies the inequalities

$$\sup_{t \in [0, 2\pi]} |z_i(t)| < R, \qquad (1 \leqslant i \leqslant n).$$

Define the linear mapping $A\colon X \to X$ by

$$Az = -\left(\frac{a_1'}{(p+1)\, a_1} z_1, ..., \frac{a_n'}{(p+1)\, a_n} z_n \right).$$

If

$$\arg a_i(2\pi) - \arg a_i(0) = 2\pi k_i, \qquad (1 \leqslant i \leqslant n),$$

it follows from the 2π-periodicity of the a_i that the k_i are integers. Also, $\ker(L-A)$ has the form

$$(b_1 a_1^{-1/(p+1)}, b_2 a_2^{-1/(p+1)}, ..., b_n a_n^{-1/(p+1)}),$$

where the $b_i \in \mathbb{C}$ and $b_i = 0$ if $k_i/(p+1) \notin \mathbb{Z}$, $(i = 1, 2, ..., n)$. Moreover [10],

$$\operatorname{Im}(L-A) = \left\{ z \in X : \int_0^{2\pi} a_i(t)^{1/(p+1)} z_i(t)\, dt = 0 \text{ whenever } \frac{k_i}{p+1} \in \mathbb{Z} \right\}.$$

Note that, in our setting, there is no ambiguity in the consideration of the roots $a_i^{-1/(p+1)}$ and $a_i^{1/(p+1)}$ as well as similar roots which appear below. We can choose for Q the projector defined by $Qz = (Q_1 z_1, ..., Q_n z_n)$ with $Q_i = 0$ if $k_i/(p+1) \notin \mathbb{Z}$, and

$$Q_i z_i = \left[\frac{\int_0^{2\pi} a_i(t)^{1/(p+1)} z_i(t)\, dt}{\int_0^{2\pi} |a_i(t)|^{2/(p+1)}\, dt} \right] \bar{a}_i^{1/(p+1)},$$

if $k_i/(p+1) \in \mathbb{Z}$. With those results, it is not diffficult to check that $L-A$ is a Fredholm mapping of index zero and that $N-A$ is $(L-A)$-completely continuous where L and N are defined in (5) and (6). By Lemma 1, the proof is complete if $k_i/(p+1) \notin \mathbb{Z}$, for all $i = 1, 2, ..., n$. If now $1 \leqslant i_1, i_2, ..., i_\ell \leqslant n$ denote the indices for which $k_i/(p+1) \in \mathbb{Z}$, then the restriction to $\ker(L-A)$ of the mapping $Q(N-A)$ is, up to isomorphisms, equal to the mapping $F: \mathbb{C}^\ell \to \mathbb{C}^\ell$, defined by $F = (F_{i_1}, ..., F_{i_\ell})$, where $F_{i_k}: \mathbb{C}^\ell \to \mathbb{C}$, $(1 \leqslant k \leqslant \ell)$ is given by

$$\begin{aligned} &F_{i_k}(b_{i_1}, ..., b_{i_\ell}) \\ &\quad = \frac{1}{2\pi} \int_0^{2\pi} \left[|a_{i_k}(t)|^{-(2p+q)/(p+1)} |b_{i_k}|^q \bar{b}_{i_k}^p + a_{i_k}(t)^{-p/(p+1)} h(t, \beta(t)) \right] dt \\ &\qquad + \frac{b_{i_k}}{2\pi(p+1)} \int_0^{2\pi} \frac{a'_{i_k}(t)}{a_{i_k}(t)}\, dt, \end{aligned}$$

with

$$\begin{aligned} &\beta_{i_j}(t) = b_{i_j} a_{i_j}(t)^{-1/(p+1)}, && (1 \leqslant j \leqslant \ell), \\ &\beta_i(t) = 0, && (i \in \{1, 2, ..., n\} \backslash \{i_1, ..., i_\ell\}). \end{aligned}$$

Define also, $G_{i_k}\colon \mathbb{C}^{\ell} \to \mathbb{C}$ by

$$G_{i_k}(b_{i_1}, ..., b_{i_\ell}; \lambda) = (1-\lambda)\left(\frac{1}{2\pi}\int_0^{2\pi} |a_{i_k}(t)|^{-(2p+q)/(p+1)}\,dt\right) |b_{i_k}|^q\,\bar{b}_{i_k}^p + \lambda F_{i_k}(b_{i_1}, ..., b_{i_\ell}), \qquad (1 \leqslant k \leqslant \ell).$$

Then,

$$\begin{aligned} &|G_{i_k}(b_{i_1}, ..., b_{i_\ell}; \lambda)\, b_{i_k}{}^p| \\ &\geqslant \left(\frac{1}{2\pi}\int_0^{2\pi} |a_{i_k}(t)|^{-(2p+q)/(p+1)}\,dt\right) |b_{i_k}|^{2p+q} \\ &\quad - \sum_{q=1}^{\ell} \alpha_{i_k i_q}\left(\frac{1}{2\pi}\int_0^{2\pi} |a_{i_k}(t)|^{-p/(p+1)}\, |a_{i_q}(t)|^{-(p+q)/(p+1)}\,dt\right) |b_{i_q}|^{p+q}\,|b_{i_k}|^p \\ &\quad -\gamma_{i_k}\left(\int_0^{2\pi} |a_{i_k}(t)|^{-p/(p+1)}\,dt\right)|b_{i_k}|^p - \left|\frac{1}{2\pi(p+1)}\int_0^{2\pi}\frac{a'_{i_k}(t)}{a_{i_k}(t)}\,dt\right| |b_{i_k}|^{p+1}, \end{aligned} \tag{13}$$

for $1 \leqslant k \leqslant \ell$. As in (11), we have, for $1 \leqslant j \neq k \leqslant \ell$,

$$\begin{aligned} &\left(\frac{1}{2\pi}\int_0^{2\pi} |a_{i_k}(t)|^{-p/(p+1)}\, |a_{i_j}(t)|^{-(p+q)/(p+1)}\,dt\right) |b_{i_j}|^{p+q}|b_{i_k}|^p \\ &\qquad \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} |a_{i_k}(t)|^{-(2p+q)/(p+1)}\,dt\right) |b_{i_k}|^{(2p+q)/(p+1)} \\ &\qquad\quad + \left(\frac{1}{2\pi}\int_0^{2\pi} |a_{i_j}(t)|^{-(2p+q)/(p+1)}\,dt\right) |b_{i_j}|^{(2p+q)/(p+1)}, \end{aligned}$$

and hence, letting

$$D_k = \frac{1}{2\pi}\int_0^{2\pi} |a_{i_k}(t)|^{-(2p+q)/(p+1)}\,dt, \qquad B_k = \frac{1}{2\pi}\int_0^{2\pi} |a_{i_k}(t)|^{-p/(p+1)}\,dt,$$

$$K_k = \left|\frac{1}{2\pi(p+1)}\int_0^{2\pi}\frac{a'_i(t)}{a_{i_k}(t)}\,dt\right|,$$

we obtain from (13) that

$$\begin{aligned} |G_{i_k}(b_{i_1}, ..., b_{i_\ell}; \lambda)\, b_{i_k}^p| \geqslant{}& \left(1 - \sum_{j=1}^{\ell} \alpha_{i_k i_j}\right) D_k |b_{i_k}|^{2p+q} - \sum_{j=1, j\neq k}^{\ell} \alpha_{i_k i_j} D_j |b_{i_j}|^{2p+q} \\ & -\gamma_{i_k} B_k |b_{i_k}|^p - K_k |b_{i_k}|^{p+1}, \qquad (1 \leqslant k \leqslant \ell). \end{aligned}$$

By the property of matrix (4), it follows that

$$\sum_{k=1}^{\ell} D_k |b_{i_k}|^{3p+q} |G_{i_k}(b_{i_1}, ..., b_{i_\ell}; \lambda)|$$
$$\geqslant \mu \sum_{k=1}^{\ell} D_k^2 |b_{i_k}|^{2(2p+q)} - \sum_{k=1}^{\ell} [\gamma_{i_k} B_k D_k |b_{i_k}|^{3p+q} + K_k D_k |b_{i_k}|^{3p+q+1}].$$

Consequently, $G(b_{i_1}, ..., b_{i_\ell}; \lambda) \neq 0$ for $|(b_{i_1}, ..., b_{i_k})|$ sufficiently large and all $\lambda \in [0, 1]$. Hence, if $R > 0$ is large enough [14],

$$\begin{aligned}\deg_B[F, B(R), 0] &= \deg_B[G(\cdot, 1), B(R), 0] \\ &= \deg_B[G(\cdot, 0), B(R), 0] = (-p)^\ell,\end{aligned}$$

and the results follow by Lemma 1. ∎

As the matrix defined in (4) tends to the identity matrix when $\sum_{i,j=1}^{n} |\alpha_{ij}|$ tends to 0, it is clear that the following corollary is true.

COROLLARY 1. *Assume that $p + q > 1$ and that*

$$\lim_{|z| \to \infty} \frac{|h(t, z)|}{|z|^{p+q}} = 0,$$

uniformly for $t \in [0, 2\pi]$. Then problem (2) has at least one solution.

Remark 1. When $n = 2$, the positive definiteness of matrix (4) is equivalent to the conditions

$$\alpha_{11} + \alpha_{12} < 1, \qquad \alpha_{21} + \alpha_{22} < 1, \qquad \frac{(\alpha_{12} + \alpha_{21})^2}{(1 - \alpha_{11} - \alpha_{12})(1 - \alpha_{21} - \alpha_{22})} < 4.$$

Remark 2. Systems of the form (2) occur if we consider space discretizations of nonlinear Schrödinger-type equations of the form

$$\alpha(t, x) \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + |z|^q \bar{z}^p + \varphi(t, x),$$

with 2π-periodic boundary conditions in t and zero Dirichlet conditions on $[a, b]$. Indeed, letting

$$z_i(t) = z\left(t, a + \frac{i(b-a)}{n}\right), \qquad f_i(t) = \varphi\left(t, a + \frac{i(b-a)}{n}\right),$$
$$a_i(t) = \alpha\left(t, a + \frac{i(b-a)}{n}\right), \qquad (1 \leqslant i \leqslant n),$$

the discretization takes the form of the system

$$a_i(t)\, z_i'(t) = \frac{n^2}{(b-a)^2}\,[z_{i+1}(t) - 2z_i(t) + z_{i-1}(t)]$$
$$+ |z_i(t)|^q\, \bar{z}_i(t)^p + f_i(t), \qquad (1 \leqslant i \leqslant n),$$

which is a special case of (2).

For a single equation, Theorem 1 takes the following simple form. Let $q \in \mathbb{R}$ be nonnegative, and $p \geqslant 1$ an integer. Let $h: [0, 2\pi] \times \mathbb{C} \to \mathbb{C}$ be a continuous mapping, and $a: \mathbb{R} \to \mathbb{C} \backslash \{0\}$ be of class C^1 and 2π-periodic.

Corollary 2. *If $p + q > 1$ and if*

$$|h(t, z)| \leqslant \alpha\, |z|^{p+q} + \beta, \tag{14}$$

for some $\alpha < 1$, $\beta \geqslant 0$ and all $t \in [0, 2\pi]$ and $z \in \mathbb{C}$, then the problem

$$a(t)\, z' = |z|^q\, \bar{z}^p + h(t, z), \qquad z(0) = z(2\pi). \tag{15}$$

has at least one solution.

The following result, which slightly extends the ones of [12, 13], is an easy consequence of Corollary 2. Let r, s be nonnegative integers, m an integer, $c \in \mathbb{C} \backslash \{0\}$, and $b_{kl}: [0, 2\pi] \to \mathbb{C}$ be continuous functions, where k and l are nonnegative integers such that $k + l < r + s$.

Corollary 3. *Assume that $r + s > 1$ and $r \leqslant s - 1$. Then the problem*

$$z' = \sum_{0 \leqslant k+l < r+s} b_{kl}(t)\, z^k \bar{z}^l + c e^{imt} z^r \bar{z}^s, \qquad z(0) = z(2\pi). \tag{16}$$

has at least one solution.

Remark 3. When $r + s = 1$ and $r \leqslant s - 1$, then $r = 0$ and $s = 1$ and (16) reduces to the linear problem

$$z' = b_{00}(t) + c e^{imt} \bar{z}, \qquad z(0) = z(2\pi).$$

When $m = 0$, it is easy to check that such a problem has a solution for each b_{00}. Such a result is not true when $m \neq 0$, as shown by the example

$$z' = i e^{2it} \bar{z} + i e^{it}, \qquad z(0) = z(2\pi),$$

which has no solution. Indeed, if z is a solution, then $u(t) = e^{-it} z(t)$ will be a 2π-periodic solution of the equation

$$u' = i(\bar{u} - u) + i,$$

so that, letting $u = v + iw$, w will be a 2π-periodic solution of the equation

$$w' = 1,$$

which is impossible. Thus the assumptions on s in Corollary 4 below are sharp.

COROLLARY 4. *If m and s are integers, with $s \geqslant 2$ when $m \neq 0$ and $s \geqslant 1$ when $m = 0$, if the functions $c_j : [0, 2\pi] \to \mathbb{C}$, $(0 \leqslant j \leqslant s-1)$ are continuous and if $c \in \mathbb{C} \setminus \{0\}$, then the problem*

$$z' = \sum_{j=0}^{s-1} c_j(t)\, \bar{z}^j + c e^{imt} \bar{z}^s, \qquad z(0) = z(2\pi),$$

has at least one solution.

3. Systems with Ginzburg–Landau Nonlinearities

Equation (16) when $r > s+1$ contains in particular the case of the Riccati equation

$$z' = z^2 + g(t)\, z + f(t),$$

which, as shown in [5, 4], may have no 2π-periodic solutions for some 2π-periodic real valued coefficients g and f. Very recently, an example with $g = 0$ and no 2π-periodic solutions has even been constructed explicitly by Campos and Ortega [2]. If $r = s$, then Eq. (16) becomes

$$z' = \sum_{0 \leqslant k+l \leqslant 2s-1} b_{kl}(t)\, z^k \bar{z}^l + c e^{imt}\, |z|^{2s}, \qquad z(0) = z(2\pi),$$

and it is easy to find, as in Section 2, *a priori* bounds for the possible 2π-periodic solutions. But the associated degree on large balls is equal to zero, so that the situation recalls the Ambrosetti–Prodi one [8]. However, the classical techniques used for this problem in the case of other equations seem to fail here and the question is still open. In the remaining case of (16) with $r = s+1$, i.e. for the problem

$$z' = \sum_{0 \leqslant k+l \leqslant 2s} b_{kl}(t)\, z^k \bar{z}^l + c e^{imt}\, |z|^{2s}\, z, \qquad z(0) = z(2\pi),$$

existence results had been given in [9] when $m = 0$, and we extend them to the more general problem (15). Notice that nonlinearities of the type of the higher order term in the above equation occur in the Ginzburg–Landau equation which arose in the theory of superconductivity and has been applied since to other problems of physics (see, e.g., [1, p. xvii, 3, p. 286]).

More generally, we consider the problem

$$a_i(t)\, z_i' = |z_i|^q\, z_i + h_i(t, z), \qquad z_i(0) = z_i(2\pi), \qquad (1 \leqslant i \leqslant n), \tag{17}$$

where q is a positive real number, $h_i : [0, 2\pi] \times \mathbb{C} \to \mathbb{C}$ is continuous and $a_i : [0, 2\pi] \to \mathbb{C}$ is continuous, $(1 \leqslant i \leqslant n)$. Our first result goes as follows.

THEOREM 2. *Assume that all the functions* $\Re a_i$ *do not vanish and have the same sign on* $[0, 2\pi]$ $(1 \leqslant i \leqslant n)$ *and that*

$$\left| \Re \left[\sum_{i=1}^{n} \frac{\overline{z_i} h_i(t, z)}{a_i(t)} \right] \right| \leqslant \alpha \left(\sum_{i=1}^{n} \frac{|\Re a_i(t)|}{|a_i(t)|^2} |z_i|^{q+2} \right) + \beta, \tag{18}$$

for some $\alpha \in [0, 1[$, *some* $\beta \geqslant 0$ *and all* $t \in [0, 2\pi]$ *and* $z \in \mathbb{C}^n$. *Then problem* (17) *has at least one solution.*

Proof. We apply Krasnosel'skii's method of guiding functions, as stated for example in Proposition VI.6 and the following Remark 1 of [7]. Let $\varepsilon = -\operatorname{sign}(\Re a_i(t))$, and $V : \mathbb{C}^n \to \mathbb{R}$ be defined by $V(z) = (\varepsilon/2) \sum_{i=1}^{n} |z_i|^2$. Thus $|V|$ is coercive and

$$\nabla V(z) = \varepsilon(\Re z_1, \Im z_1, \ldots, \Re z_n, \Im z_n) = \varepsilon(z_1, \ldots, z_n) = \varepsilon z.$$

Moreover, if

$$(u, v) = \sum_{i=1}^{n} (\Re u_i \Re v_i + \Im u_i \Im v_i) = \Re \left[\sum_{i=1}^{n} (u_i \overline{v_i}) \right] = \Re \left[\sum_{i=1}^{n} (\overline{u_i} v_i) \right],$$

denotes the usual inner product in $\mathbb{C}^n$, and if we set

$$f(t, z) = \left(\frac{|z_1|^q\, z_1 + h_1(t, z)}{a_1(t)}, \ldots, \frac{|z_n|^q\, z_n + h_n(t, z)}{a_n(t)} \right),$$

we have, using (18),

$$\begin{aligned}(\nabla V(z), f(t, z)) &= \varepsilon \Re \left[\sum_{i=1}^{n} \overline{z_i} \left(\frac{|z_i|^q\, z_i + h_i(t, z)}{a_i(t)} \right) \right] \\ &= -\sum_{j=1}^{n} \frac{|\Re a_i(t)|}{|a_i(t)|^2} |z_i|^{q+2} + \varepsilon \Re \left[\sum_{i=1}^{n} \frac{\overline{z_i} h_i(t, z)}{a_i(t)} \right]\end{aligned}$$

$$\leqslant (-1+\alpha) \sum_{j=1}^{n} \frac{|\Re a_i(t)|}{|a_i(t)|^2} |z_i|^{q+2} + \beta$$

$$\leqslant -(1-\alpha) \left[\max_{1 \leqslant i \leqslant n} \max_{t \in [0, 2\pi]} \frac{|\Re a_i(t)|}{|a_i(t)|^2} \right] \left(\sum_{j=1}^{n} |z_j|^{q+2} \right) + \beta$$

$$\leqslant -(1-\alpha) \left[\max_{1 \leqslant i \leqslant n} \max_{t \in [0, 2\pi]} \frac{|\Re a_i(t)|}{|a_i(t)|^2} \right] n^{-q/2} |z|^{q+2} + \beta.$$

Consequently, there exists $R > 0$ such that $(\nabla V(z), f(t, z)) < 0$ whenever $|z| \geqslant R$. Thus all conditions of Krasnosel'skii's theorem hold. ∎

In the case of the single equation

$$a(t)\, z' = |z|^q\, z + h(t, z), \qquad z(0) = z(2\pi), \tag{19}$$

with $a : [0, 2\pi] \to \mathbb{C}$ and $h : [0, 2\pi] \times \mathbb{C}$ continuous, Theorem 2 directly implies the following existence result.

COROLLARY 5. *Assume that the function* $\Re a$ *does not vanish and that*

$$\left| \Re \left[\frac{\bar{z} h(t, z)}{a(t)} \right] \right| \leqslant \alpha \left(\frac{|\Re a(t)|}{|a(t)|^2} |z|^{q+2} \right) + \beta, \tag{20}$$

for some $\alpha \in [0, 1[$, *some* $\beta \geqslant 0$ *and all* $t \in [0, 2\pi]$ *and* $z \in \mathbb{C}$. *Then problem* (19) *has at least one solution.*

Notice that condition (20) holds in particular when

$$|h(t, z)| \leqslant \alpha \frac{|\Re a(t)|}{|a(t)|} |z|^{q+1} + \beta,$$

for some $\alpha \in [0, 1[$, some $\beta \geqslant 0$ and all $t \in [0, 2\pi]$ and $z \in \mathbb{C}$. Corollary 5 can be applied to problem (16) with $r = s + 1$ and $m = 0$. The conditions then reduce to $\Re c \neq 0$ and, together with Theorem 1, we obtain a complete extension of Theorem 1 of [9] to problem (16). A situation not covered by this last theorem and which is handled by Corollary 5 is the problem

$$z' = \sum_{0 \leqslant k + l \leqslant 2s} b_{kl}(t)\, z^k \bar{z}^l + (b + e^{imt})\, |z|^{2s}\, z, \qquad z(0) = z(T),$$

for any real number b such that $|b| > 1$.

Remark 4. In contrast with the case of Eq. (15), Theorem 2 remains valid for $q=0$. In this case, a simple application of Schauder's fixed point theorem shows that a solution of (19) with $q=0$ exists when

$$\lim_{|z| \to \infty} \frac{h(t, z)}{|z|} = 0,$$

uniformly in $t \in [0, 2\pi]$, $a(t) \neq 0$ for all $t \in [0, 2\pi]$ and $a(t)\, z' = z$ has only the trivial 2π-periodic solution. This is in particular the case when

$$\int_0^{2\pi} \frac{\Re a(t)}{|a(t)|^2}\, dt \neq 0.$$

It is then natural to raise the question of obtaining similar results for the periodic solutions of systems with Ginzburg–Landau nonlinearities when the functions $\Re a_i$ do not have the same sign for all $1 \leqslant i \leqslant n$. Then the method of guiding functions does not seem to apply and a possibility is to try to find conditions under which the possible solutions of the family of problems

$$z_i' = (1-\lambda) \frac{R_i^q}{a_i(t)} z_i + \lambda \left[\frac{|z_i|^q z_i + h_i(t, z)}{a_i(t)} \right], \qquad \lambda \in [0, 1],$$

$$z_i(0) = z_i(2\pi), \qquad (1 \leqslant i \leqslant n),$$

will have no solution on the boundary of the open bounded set

$$\Omega = \{ z \in X : |z_i(t)|^2 < R_i^2,\ t \in [0, 2\pi],\ 1 \leqslant i \leqslant n \},$$

for sufficiently large numbers R_i. Maximum principle type arguments can make this process successful but seem to require some rather weak coupling between the function h_i. Hence, at the expense of a stronger regularity condition upon the a_i, we show that the methodology of Section 2 provides existence conditions for a larger class of functions h_i.

THEOREM 3. *Assume that the functions $a_i : [0, 2\pi] \to \mathbb{C}$ are of class C^1, that $\Re a_i(t) \neq 0$ for all $t \in [0, 2\pi]$ and that the functions $|a_i|^2/2\Re a_i$ are 2π-periodic $(1 \leqslant i \leqslant n)$. Assume moreover that*

$$\lim_{|z| \to \infty} \frac{\Re[\bar{a}_i(t)\, h_i(t, z)\, \bar{z}_i]}{|z_i|\, |z|^{q+1}} = 0, \tag{21}$$

uniformly in $t \in [0, 2\pi]$. Then problem (17) *has at least one solution.*

Proof. Like in the proof of Theorem 1, we use a continuation argument based upon Lemma 1 and first determine continuous functions $b_i : \mathbb{R} \to \mathbb{C}$ $(1 \leqslant i \leqslant n)$ such that an a priori bound exists for the possible solutions of the one-parameter family of problems

$$a_i(t)\, z_i' + b_i(t)\, z_i = \lambda[b_i(t)\, z_i + h_i(t, z) + |z_i|^q\, z_i], \qquad \lambda \in\,]0, 1], \tag{22}$$
$$z_i(0) = z_i(2\pi), \qquad (1 \leqslant i \leqslant n).$$

If z is a possible solution of (22), then we have also, for each $\lambda \in\,]0, 1]$,

$$\overline{a_i}(t)\, \overline{z_i}' + \overline{b_i}(t)\, \overline{z_i} = \lambda[\overline{b_i}(t)\, \overline{z_i} + \overline{h_i}(t, z) + |z_i|^q\, \overline{z_i}], \qquad z_i(0) = z_i(2\pi), \tag{23}$$

for each $1 \leqslant i \leqslant n$. Hence, multiplying (2) by $\overline{a_i}(t)\, \overline{z_i}$, (23) by $a_i(t)\, z_i$ and adding the results, we get

$$|a_i(t)|^2 \frac{d}{dt}(|z_i(t)|^2) + 2\mathfrak{R}[\overline{a_i}(t)\, b_i(t)]|z_i(t)|^2$$
$$= \lambda\{2\mathfrak{R}[\overline{a_i}(t)\, b_i(t)]|z_i(t)|^2$$
$$+ 2\mathfrak{R}[\overline{a_i}(t)\, h_i(t, z(t))\, \overline{z_i}(t)] + 2\mathfrak{R}a_i(t)|z_i(t)|^{q+2}\},$$

or, equivalently, introducing the real 2π-periodic functions $c_i = |a_i|^2/2\mathfrak{R}a_i$, we have, for each $1 \leqslant i \leqslant n$,

$$\frac{d}{dt}[c_i(t)|z_i(t)|^2] + \left[\frac{\mathfrak{R}(\overline{a_i}(t)\, b_i(t))}{\mathfrak{R}a_i(t)} - c_i'(t)\right] |z_i(t)|^2$$
$$= \lambda\left[\frac{\mathfrak{R}(\overline{a_i}(t)\, b_i(t))}{\mathfrak{R}a_i(t)} |z_i(t)|^2 + \frac{\mathfrak{R}[\overline{a_i}(t)\, h_i(t, z(t))\, \overline{z_i}(t)]}{\mathfrak{R}a_i(t)} + |z_i(t)|^{q+2}\right]. \tag{24}$$

We now choose b_i such that

$$\frac{\mathfrak{R}(\overline{a_i}(t)\, b_i(t))}{\mathfrak{R}a_i(t)} = c_i'(t),$$

i.e., such that

$$(a_i(t), b_i(t)) = c_i'(t)\, \mathfrak{R}a_i(t), \tag{25}$$

where the left hand member denotes the usual inner product in $\mathbb{C}$ defined by

$$(a_i(t), b_i(t)) = \mathfrak{R}a_i(t)\, \mathfrak{R}b_i(t) + \mathscr{I}a_i(t)\, \mathscr{I}b_i(t) = \mathfrak{R}[\overline{a_i}(t)\, b_i(t)].$$

A solution of (25) is given by

$$b_i(t) = \frac{c_i'(t)\,\Re a_i(t)}{|a_i(t)|^2}\,a_i(t) = \frac{c_i'(t)}{2c_i(t)}\,a_i(t). \tag{26}$$

With this choice of b_i $(1 \leqslant i \leqslant n)$, it follows from (24), integrating both members from 0 to 2π and dividing by $\lambda > 0$, that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} &|z_i(t)|^{q+2}\,dt\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\Re[\bar{a}(t)\,h_i(t, z(t))\,\overline{z}_i(t)]}{\Re a_i(t)}\right| dt + \frac{1}{2\pi}\int_0^{2\pi} |c_i'(t)|\,|z_i(t)|^2\,dt.\end{aligned}$$

Now, it follows from (21) that, for any $\alpha_i > 0$, we can find $\gamma_i \geqslant 0$ such that

$$\left|\frac{\Re[\overline{a}_i(t)\,h_i(t, z)\,\overline{z}_i(t)]}{\Re a_i(t)}\right| \leqslant \alpha_i |z_i| \left(\sum_{j=1}^{n} |z_j|^{q+1}\right) + \gamma_i,$$

for all $t \in [0, 2\pi]$ and $z \in \mathbb{C}^n$. Hence we obtain the inequality

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |z_i(t)|^{q+2}\,dt \leqslant \sum_{j=1}^{n} \frac{\alpha_i}{2\pi}\int_0^{2\pi} &|z_i(t)|\,|z_j(t)|^{q+1}\,dt\\ &+ \frac{1}{2\pi}\int_0^{2\pi} |c_i'(t)|\,|z_i(t)|^2\,dt + \gamma_i.\end{aligned} \tag{27}$$

Then, letting

$$\|z_i\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |z_i(t)|^{q+2}\,dt\right)^{1/(q+2)}, \qquad (1 \leqslant i \leqslant n),$$

using Hölder inequality and (11), we deduce from (27) that, for each $1 \leqslant i \leqslant n$,

$$\|z_i\|^{q+2} \leqslant n\alpha_i \|z_i\|^{q+2} + \alpha_i \sum_{1 \leqslant j \leqslant n,\, j \neq i} \|z_j\|^{q+2} + \eta_i \|z_i\|^2 + \gamma_i, \tag{28}$$

where $\eta_i = \max_{t \in [0, 2\pi]} |c_i'(t)|$. Taking the α_i sufficiently small so that the matrix

$$\begin{pmatrix} 1 - n\alpha_1 & -\alpha_1 & -\alpha_1 & \ldots & -\alpha_1 \\ -\alpha_2 & 1 - \mathrm{n}\alpha_2 & -\alpha_2 & \ldots & -\alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n & -\alpha_n & -\alpha_n & \ldots & 1 - n\alpha_n \end{pmatrix} \tag{29}$$

is positive definite, we find a $\mu > 0$ such that

$$\mu \sum_{i=1}^{n} \|z_i\|^{2(q+2)} \leqslant \sum_{i=1}^{n} (\eta_i \|z_i\|^{q+4} + \gamma_i \|z_i\|^{q+2}),$$

which, implies, as in the proof of Theorem 1, that

$$\sum_{i=1}^{n} \|z_i\|^{2(q+2)} < R_1,$$

for some $R_1 > 0$, and that an *a priori* bound exists for z_i in the uniform norm $(1 \leqslant i \leqslant n)$. For $\lambda = 0$, problem (22) reduces to

$$a_i(t)\, z_i' + \frac{c_i'(t)}{2c_i(t)}\, a_i(t)\, z_i = 0, \qquad (1 \leqslant i \leqslant n),$$

admits the n-parameter family of 2π-periodic solutions

$$z_i(t) = d_i |c_i(t)|^{-1/2}, \qquad d_i \in \mathbb{C}, \quad (1 \leqslant i \leqslant n),$$

and its adjoint equation has the solutions

$$z_i(t) = d_i |c_i(t)|^{1/2}, \qquad d_i \in \mathbb{C}, \quad (1 \leqslant i \leqslant n),$$

as directly checked. The corresponding mapping F as introduced in the proof of Theorem 1 is easily seen to be the mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined by

$$\begin{aligned} F_i(d) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{a_i(t)} \big[&|c_i(t)|^{-q/2} |d_i|^q\, d_i + b_i(t)\, d_i \\ &+ |c_i(t)|^{1/2}\, h_i(t, |c_1(t)|^{-1/2}\, d_1, ..., |c_n(t)|^{-1/2}\, d_n) \big]\, dt, \qquad (1 \leqslant i \leqslant n). \end{aligned}$$

Proceeding like in the proof of Theorem 1, one can show that $F(d) \neq 0$ for sufficiently large $|d|$ and that $|\deg_B[F, B(R), 0]| = 1$ for sufficiently large R. The result then follows from Lemma 1, and the proof is complete. ∎

COROLLARY 6. *Assume that the functions $a_i : [0, 2\pi] \to \mathbb{C}$ are of class C^1, that $\Re a_i(t) \neq 0$ for all $t \in [0, 2\pi]$ and that the functions $|a_i|^2 / 2\Re a_i$ are 2π-periodic $(1 \leqslant i \leqslant n)$. Assume moreover that*

$$\lim_{|z| \to \infty} \frac{h(t, z)}{|z|^{q+1}} = 0,$$

uniformly in $t \in [0, 2\pi]$. Then problem (17) has at least one solution.

Remark 5. Systems of the form (17) occur as spatial discretizations of Ginzburg–Landau partial differential equations of the type

$$\alpha(t, x)\,\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + |z|^q\, z + \varphi(t, x).$$

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